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L.J. Laslett, S. Caspi, and M. Helm

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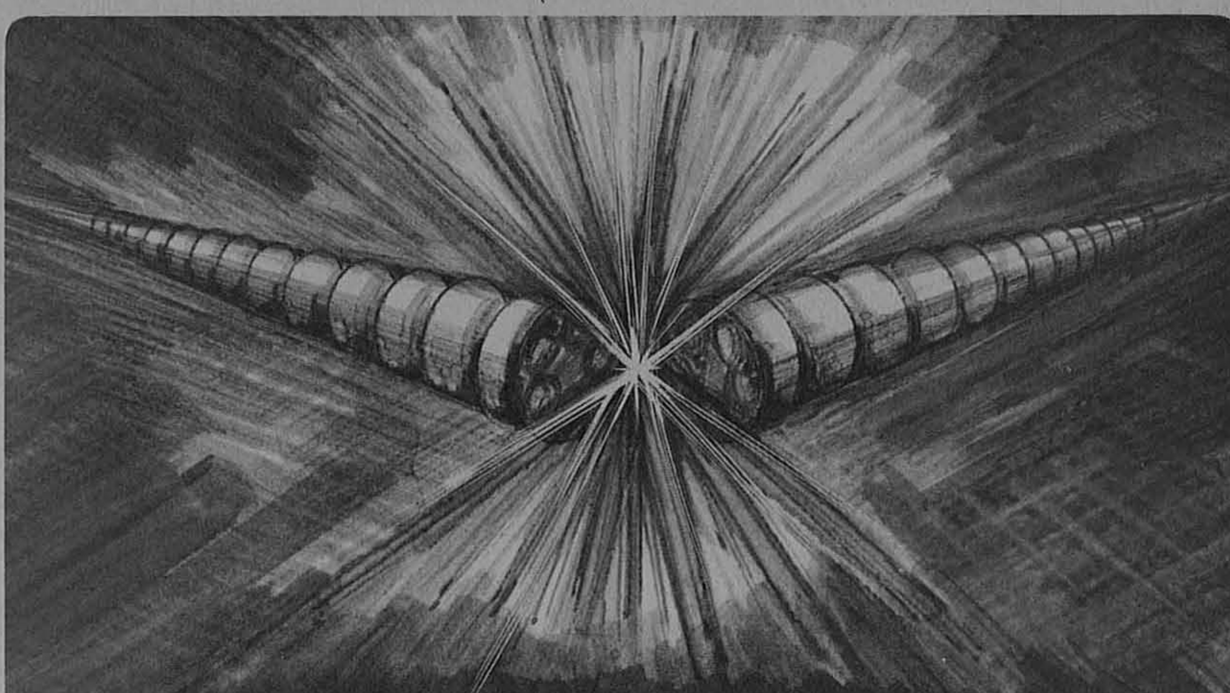

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ABSTRACT

Configurations are proposed, and illustrated, for terminating the "cosine ϕ" windings of an ideal iron-free dipole magnet so as to preserve the quality of the internal field integrated (vs. z) through the entire magnet. The end windings are placed on a surface of the same radius as that on which the conductors lie in the central (2-D) portion of the structure. The desired pure dipole quality of the integrated field then is assured by requiring that the z-component of current, after projection onto the y-z plane, shall have in that plane a density distribution whose integral is independent of y.

As a result of the analysis, end-winding configurations that satisfy this requirement are proposed in which each conductor filament follows through the transition region a locus whose y-z projection is of the form $z(y;y_0) = f(y_0) - f(y-y_0)$, with $f(0) = 0$, wherein y_0 serves as an index to identify the location of the filament in the central (2-D) portion of the structure.

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Simple solutions of this nature are indicated in which the function f has the form $f(y-y_o) = k \left(\frac{y-y_o}{a}\right)^p$ for windings on the surface of a cylinder of radius a, with $p < 1$ (and preferably $p \leq ½$) to ensure a smooth transition into the end region. The straightforward extension of these results to configurations for the production of integrated fields of higher multipolarity also is indicated.

<u>CONTENTS</u>

I. Introduction

Two-Dimensional magnet designs are well known in which at any radius the density of axial current ideally is continuously proportional to the cosine of a multiple (m) of the azimuth angle. The resulting internal magnetic field then is characterized by a z-directed vector potential that likewise is directly proportional to cos $m\phi$. (Thus, with m=1 we obtain a constant, y-directed magnetic field; for m=2, a pure quadrupole field; etc.) The issue of interest here concerns the ways in which the windings may be terminated at the ends of such a 2-D design in order that the <u>integrated</u> field (integrated through and beyond the full 3-D structure) shall retain the desired harmonic purity. It is the intention that, in this examination, the termination windings shall be restricted to the radius (or radii) that they individually occupy in the 2-D portion of the design. Under these conditions the magnetostatic problem for the <u>integrated</u> field becomes reduced to a 2-D problem in which the boundary condition at the winding radius is given explicitly in terms solely of the longitudinal integral of the current (presuming we may assume the longitudinal invariance of any surrounding highly-permeable ferromagnetic shell that may be present).

It will be recognized that the windings of typical large magnet structures, including some superconducting designs now under consideration, may be formed in practice from a limited number of current blocks (of different angular extent), separated by wedges (each of suitable angular extent) that carry no current. Adjustment of the available angular parameters of such a discrete design then can serve to suppress to zero (or to acceptably small values) a certain number of undesired Fourier components of the current distribution, and thus correspondingly suppress the associated multipole harmonics in the integrated magnetic field. The means for suitably terminating the individual winding blocks of such a discrete design can be guided only qualitatively by the analysis to be presented in the present report, but a multi-block design can benefit from the opportunity to commence the termination of the separate blocks individually at suitably chosen distinct longitudinal locations. The treatment of continuous winding distributions, as considered specifically in the present report, thus in practice may find its direct application chiefly in the design of terminations for correction windings (e.g., to the design of short-circuited, superconducting, self-correction coils, or possibly in the use of printed-circuit techniques). Such applications then may most frequently relate to windings of higher than dipole (or quadrupole) multipolarity.

II. Method

We shall commence with an examination of the manner in which one may undertake the termination of "cos ϕ" windings, so as to preserve the pure dipole character of the integrated internal field. We accordingly require that the longitudinal integral of the current shall have the same azimuthal dependence as the ideal azimuthal distribution employed in the 2-D portion of the design. It is convenient in this case to consider explicitly the character of suitable solutions as described by the locations of filament loci in the end regions when viewed in a transverse projection onto the y-z plane.

In such a y-z projection, we require that the z-component of current shall have in that plane a density distribution whose integral is independent of y, since the factor $d(a\phi)/dy = 1/\cos\phi$ transforms the reference cos ϕ distribution into a constant density with respect to the projected coördinate $y=a \sin\phi$. Solutions for such projected loci, $z(y;y_0)$, for windings situated on a radius a in the end regions, are discussed in the following Section. Such solutions then can be equivalently expressed, if desired, in terms of an alternative projection of the current filaments onto a "developed" (or w-z) view. As will be noted later, solutions expressed in this latter form can conveniently be generalized to provide analogous means for terminating the cos $m\phi$ windings of structures designed to provide fields of higher multipolarity.

III. Termination of the cos ϕ Windings of Dipole Magnets

A. Analysis

In the straight (2-D) portion of a "cos ϕ" winding, $dI/d(a\phi) = J_{z_0} \cos\phi$ with J_{z_0} = constant. With $y = a \sin\phi$, the z-component of current density when viewed in the y-z plane then assumes the constant value $dI/dy = J_{z_0}$, independent of y, in the 2-D region. As we have noted previously for the termination of cos ϕ windings, the current filaments accordingly shall be so disposed in the end regions that in a y-z projection the z-component of current shall have in that plane a density distribution whose integral, vs. z at constant y, is independent of y. The loci of such end windings may be described in this projection by a function $z(y;y_0)$, where y_0 serves as an indexing parameter to denote the value of $y=a \sin\phi$ at which the filament under consideration originates in the 2-D portion of the assembly. Such a filament departs from the straight region at $z_s(y_0) = z(y_0;y_0)$. The integration of $J_{z(\text{Projected})}$ extends in the transition region at constant y between $z_F(y) = z(y;0)$ and $z_s(y)$, and continues in the straight region to some reference location z_p (Fig. 1).

We thus require that

$$\int_{z_F}^{z_S} \underbrace{J_{z(\text{Projected})}}_{\text{at y const.}} dz + (z_p - z_S)J_{z_0} \quad \text{shall be constant (independent of y)} \;.$$

By noting that at y=0 we may make the identification $z_F = z_s = 0$, we then can identify the constant mentioned above as having the value $z_p J_{z_0}$. As a result we may write the requirement in the simple form

$$\int_{z_F}^{z_S} \underbrace{J_{z(\text{Projected})}}_{\text{at y const.}} dz = z_S J_{z_0}, \quad \text{identically in y.}$$

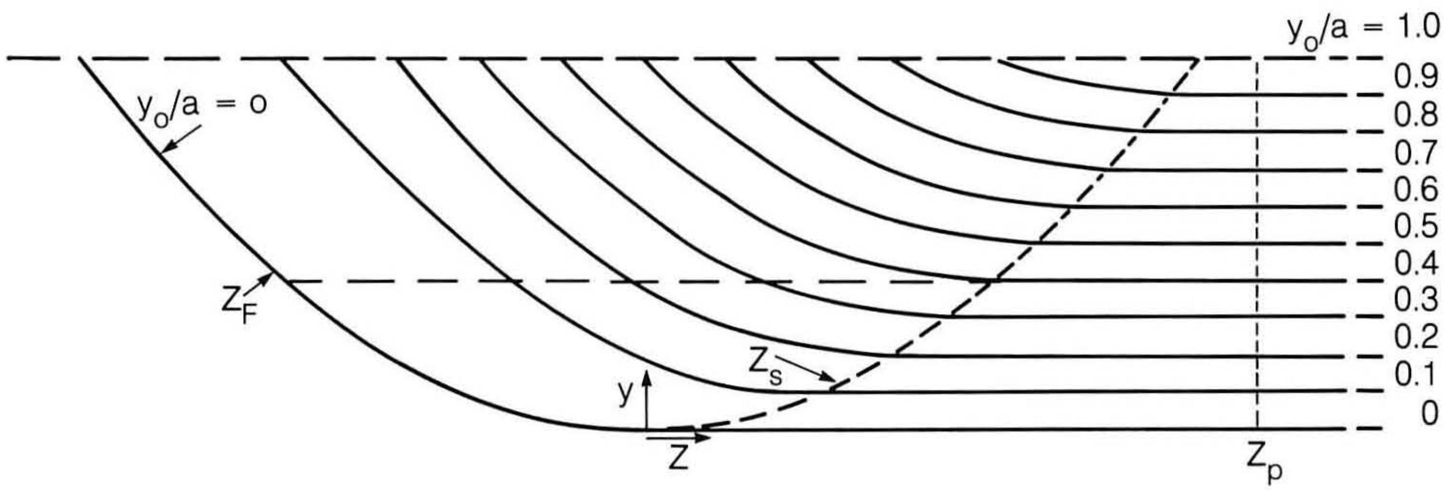

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Fig. 1. y-z projection of an illustrative termination for an integrated dipole field. The transition region lies between z_F and z_s, and within that region the loci of individual current filaments are represented by $z(y;y_0)$. A representative path of integration at constant y is illustrated here for $y/a = 0.4$. The path of integration extends from $z_F(y)$ (corresponding to $y_0/a=0$), through $z_s(y)$ (corresponding to $y_0=y$), and extends to the reference plane $z=z_p$. The filament loci shown here were drawn specifically for a type of transition considered in detail in a subsequent section (§ III. B.2), with $p = ½$ and $k = 1.5$ a.

For the integration at constant y through the transition region, the value of $J_{z(\text{Projected})}$ differs from the constant projected value in the 2-D region by the factor $\partial y_0/\partial y\big|_{z\ \text{const.}}$. The requirement for an integrated dipole field thus becomes expressed explicitly as

$$\int_{z_F}^{z_S} \underbrace{\frac{\partial y_0}{\partial y}\bigg|_{z\ \text{const.}}}_{\text{at } y\ \text{const.}} dz = z_S, \quad \text{identically in } y.$$

(We note that in the course of the integration, at a constant value of y, the value of y_0 at which the integrand is evaluated ranges from $y_0=0$ to $y_0=y$.) It is convenient to rewrite the requirement exhibited above in terms of an integral in which y_0 serves as the variable of integration. We note that

$$\frac{\partial y_0}{\partial y}\bigg|_{z\ \text{const.}} = -\left[\frac{\partial z}{\partial y}\bigg|_{y_0\ \text{const.}}\right] \bigg/ \left[\frac{\partial z}{\partial y_0}\bigg|_{y\ \text{const.}}\right],$$

so that the requirement of interest becomes

$$\int_{y_0=0}^{y_0=y} \left[-\frac{\partial z}{\partial y}\bigg|_{y_0\ \text{const.}}\right] dy_0 = z_S, \quad \text{identically in } y.$$

We now propose a solution of the form $z(y;y_0) = z_s(y_0) - f(y-y_0)$ for any particular filament (characterized by the "index" y_0) in the transition region. We shall take f(0) to be zero in the interests of continuity and, to avoid "kinks," it will be desirable that $1/f'(0) = 0$. For a function of this form, $-\frac{\partial z}{\partial y}\Big|_{y_0\ \text{const.}} = f'(y-y_0)$ and the integral relation written at the conclusion of the preceding paragraph becomes

$$\int_{y_0=0}^{y_0=y} f'(y-y_0)\, dy_0 = z_s,$$

$$- f(y-y_0) \Big|_{y_0=0}^{y_0=y} = z_s,$$

or $$f(y) = z_s.$$

We have thus only to require that $z_s(y) = f(y)$ [or, correspondingly, $z_s(y_0) = f(y_0)$] and write the solution as

$$z(y;y_0) = f(y_0) - f(y-y_0).$$

The transition region is bounded by

$$z_s(y) = f(y)$$

and by
$$\begin{aligned} z_F(y) &= z(y;0) \\ &= - f(y) \\ &= - z_s(y) \end{aligned}$$

-- resulting in boundary curves that are symmetrically situated about z=0.

B. Examples

For solutions of the type suggested in the preceding subsection, one may choose the function f with considerable freedom (but subject to the constraint f(0)=0). The transition region at y then lies between the symmetrical limits $z_F(y) = -z_s(y)$ and $z_s(y)$, where $z_s(y) = f(y)$. Within that region individual current filaments characterized by an index parameter y_0, are described by $z(y;y_0) = f(y_0) - f(y-y_0)$.

We present below several examples of such solutions for providing an integrated internal field of a pure dipole character. In each such case, the description of the specific termination is initially introduced with respect to the y-z projection of filament loci, as specified by the function $z(y;y_0) = f(y_0) - f(y-y_0)$. For each case, it is possible and informative, also to specify transition loci in a "developed" (w-z) plane. By this development we mean a development of the cylindrical surface r=a (on which the windings lie), centered about the polar location $\phi=\frac{\pi}{2}$ or y=a, so that with

$$y = a \sin \phi$$

$$w = a \left(\frac{\pi}{2} - \phi\right)$$

or

$$\cos w/a = y/a$$

filament loci in the transition region may be described by

$$z = f\left(a \cos \frac{w_0}{a}\right) - f\left(a \cos \frac{w}{a} - a \cos \frac{w_0}{a}\right) .$$

The development associated with transformation from circumferential location $a\phi$ on the circular cylinder to the developed plane is distortion free, since $\left| \frac{dw}{d(a\phi)} \right| = 1.$

1. **The Lambertson-Coupland Termination**

This termination, although characterized by a distinct kink imposed on the windings at the onset of the termination, is of a simple character that may be familiar from work already cited elsewhere.* In this example, the function f is such that $z_s(y) = \frac{y}{\tan \alpha}$ and

$$z(y;y_0) = \frac{y_0}{\tan \alpha} - \frac{y-y_0}{\tan \alpha}$$

$$= \frac{2y_0 - y}{\tan \alpha} \qquad \text{(Fig. 2a).}$$

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*The type of termination suggested by G. R. Lambertson arose in consideration of possible magnet designs for the ESCAR project at this laboratory. Reference to this suggestion, and to a related Rutherford Laboratory Report [RHEL/R 203 (1970)] of J. H. Coupland, has been made by F. E. Mills and G. H. Morgan, "A Flux Theorem for the Design of Magnet Coil Ends," in Particle Accelerators 5, 227-235 (1973).

(That this disposition of windings satisfies the desired condition for the integral of J_z is evident directly from the y-z projection of Fig. 2a; in that projection the z-component of current density within the transition region becomes modified by the additional factor ½, which is precisely compensated by the additional longitudinal interval required for the z integration at constant y.)

The equation for filament loci in the developed w-z plane is obtained immediately by the substitution $y = a \cos \frac{w}{a}$ to become

$$z = \frac{(2 \cos \frac{w_0}{a} - \cos \frac{w}{a})\, a}{\tan \alpha}$$

(Fig. 2b), with the limiting locus (for a filament originating virtually in the median plane, ϕ=0, and shown by a dashed line on Fig. 2b) described in that region by $z_F = \dfrac{a \cos \frac{w}{a}}{\tan \alpha}$.

2. **Terminations Employing the Function $f(y) = k(y/a)^p$**

Terminations of the type defined by

$$z_s(y) = f(y) = k(y/a)^p,$$

with

$$z(y;y_0) = k \left[(\frac{y_0}{a})^p - (\frac{y-y_0}{a})^p \right] ,$$

include, as a special case (p=1), the Lambertson-Coupland termination described in the preceding subsection. By imposition of the restriction $0 < p < 1$, one can avoid the discontinuity of slope that in the Lambertson-Coupland termination was experienced by current filaments at the point of entry into the transition region. Although, in the work that follows, we shall make some reference to cases for which the index p is close to unity, it will be recognized that it can be advisable to restrict this

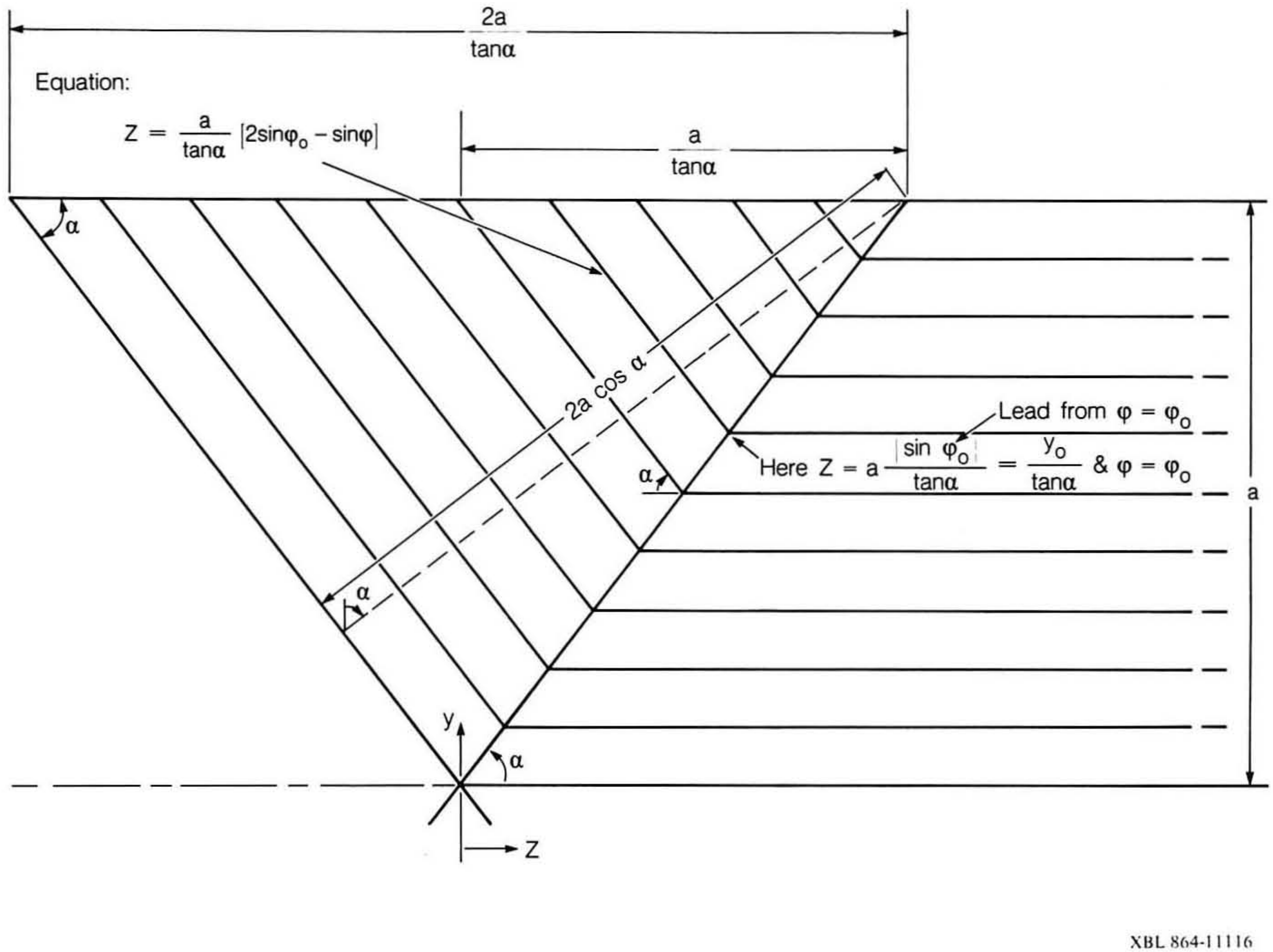

Fig. 2a. Lateral (y-z) projection of current windings situated on the surface of a circular cylinder of radius a, for the Lambertson-Coupland termination (drawn for α = 52.5°). The reference surface-current density of z-directed currents is proportional to cos ϕ on the surface of the cylinder; such currents appear to have a <u>constant</u> density in this projection, since $d(a \sin \phi)/d(a\phi) = \cos \phi$.

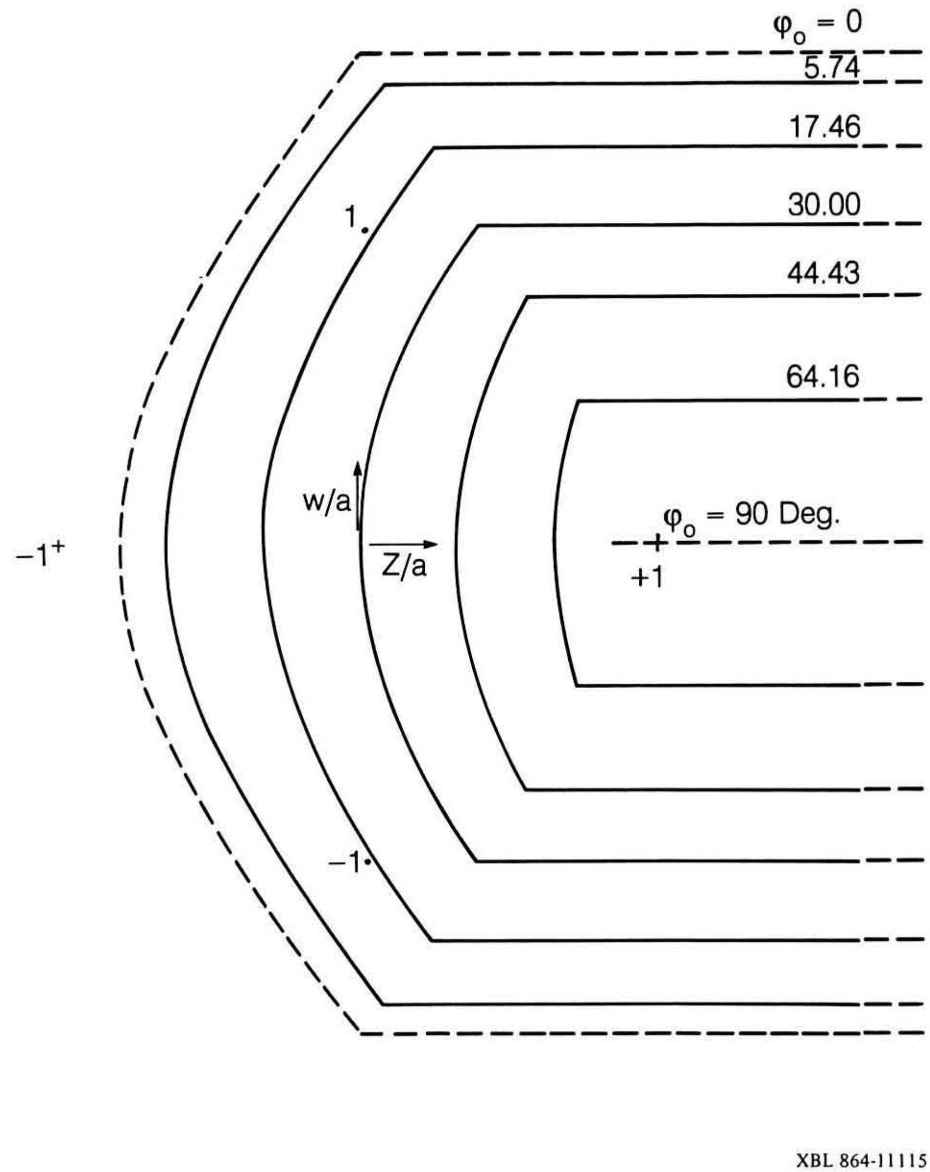

Fig. 2b. Developed (w-z) projection of some current windings for the Lambertson-Coupland termination (drawn for α = 52.5°).

index to the range $0 < p \leq ½$ in order to avoid an infinite <u>curvature</u> for the filament loci immediately upon their entry into the transition region.*

The y-z projection shown in Fig. 1 illustrates a termination of the type considered here, with $p = ½$ and $k = 1.5\ a$. The substitution $\cos w/a = y/a$ leads to the specifications for constructing the developed (w-z) view (Fig. 3):

$$z_S = - z_F = k(\cos w/a)^{½}$$

$$z = k \left[(\cos \frac{w_O}{a})^{½} - (\cos \frac{w}{a} - \cos \frac{w_O}{a})^{½} \right] .$$

It is seen from Fig. 3 that the case illustrated there is such that the boundaries of the transition region together form a nearly circular curve - - as a result of the choice $k/a = 1.5$ ($\cong \frac{\pi}{2}$).

For the type of termination considered in the present subsection, there is no choice of the parameters (p and k/a) that will result in the boundaries of the transition region having a precisely circular form in a developed view. We will demonstrate in the following subsection, however, that a design in which these boundaries do have a circular form can be constructed, if desired, which differs only slightly from a design of the present type (with $p = ½$ and $k/a = \frac{\pi}{2}$).

To conclude the present discussion of terminations defined by $z_S(y) = f(y) = k(y/a)^p$, we present a sequence of diagrams to illustrate such terminations for $p = 0.25$, $p = 0.50$, $p = 0.75$, and $p = 1.00$. Such figures show, for each case, (a) a side projection (Fig. 4, y <u>vs</u>. z), (b) a top projection (Fig. 5,

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*With $-\Delta z \propto (\Delta y)^p$, we can write $\Delta y \propto (-\Delta z)^{1/p}$ and $(\Delta y)' \propto (-\Delta z)^{1/p-1}$, so that the restriction $0 < p < 1$ suffices to ensure that $(\Delta y)' \rightarrow 0$ as $|\Delta z| \rightarrow 0$. Because $(\Delta y)'' \propto (-\Delta z)^{1/p-2}$, however, the further restriction $p \leq ½$ appears desirable so that the curvature will remain finite as $|\Delta z| \rightarrow 0$.

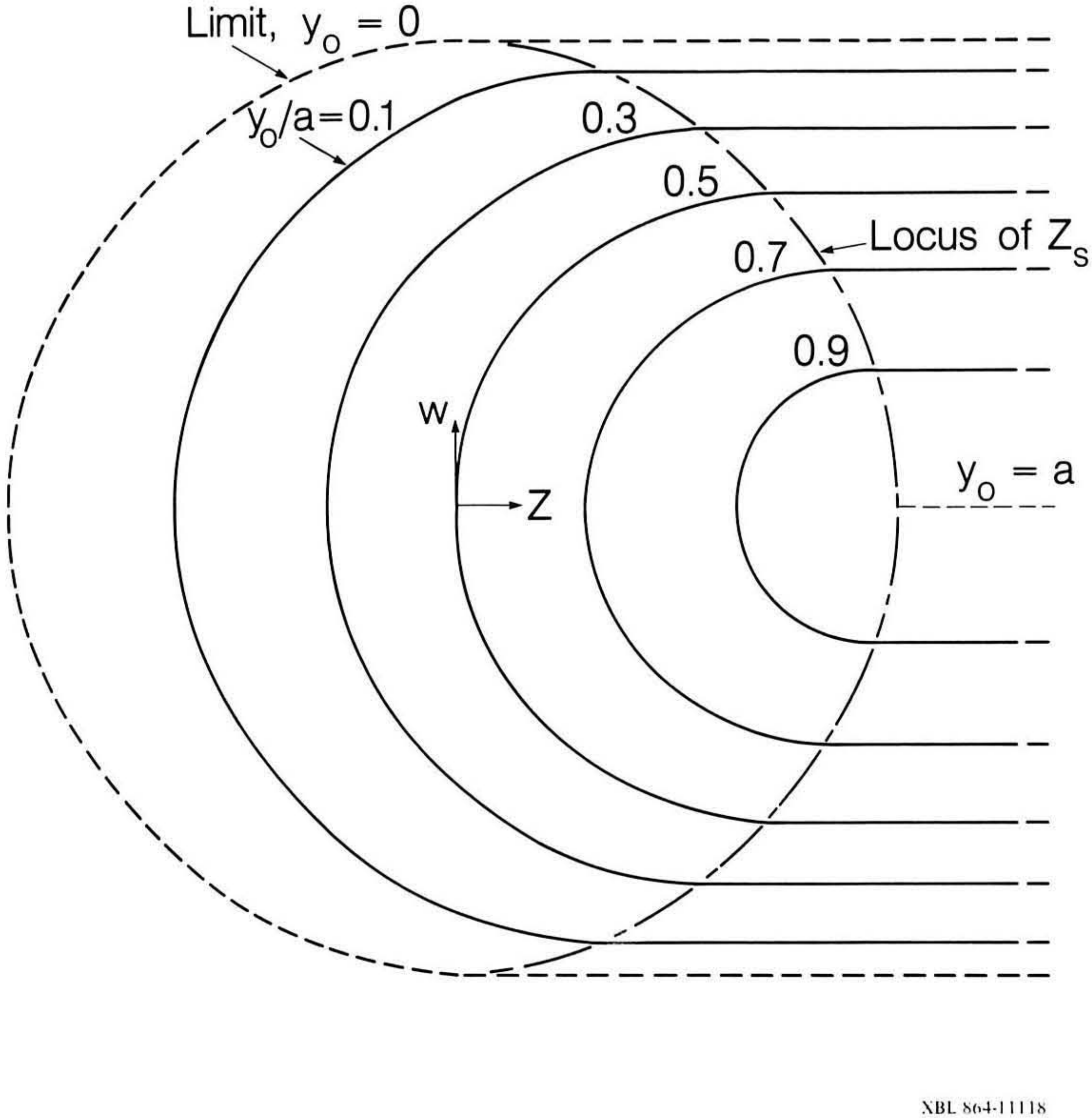

Fig. 3. Developed view (w _vs_. z) of the termination for which a y-z projection was shown as Fig. 1. $z_S(y) = -z_F(y) = f(y) = k(y/a)^{1/2}$, with k = 1.5 a, while cos w/a = y/a. The boundaries of the transition region for this case form a nearly circular curve in this developed view (see text).

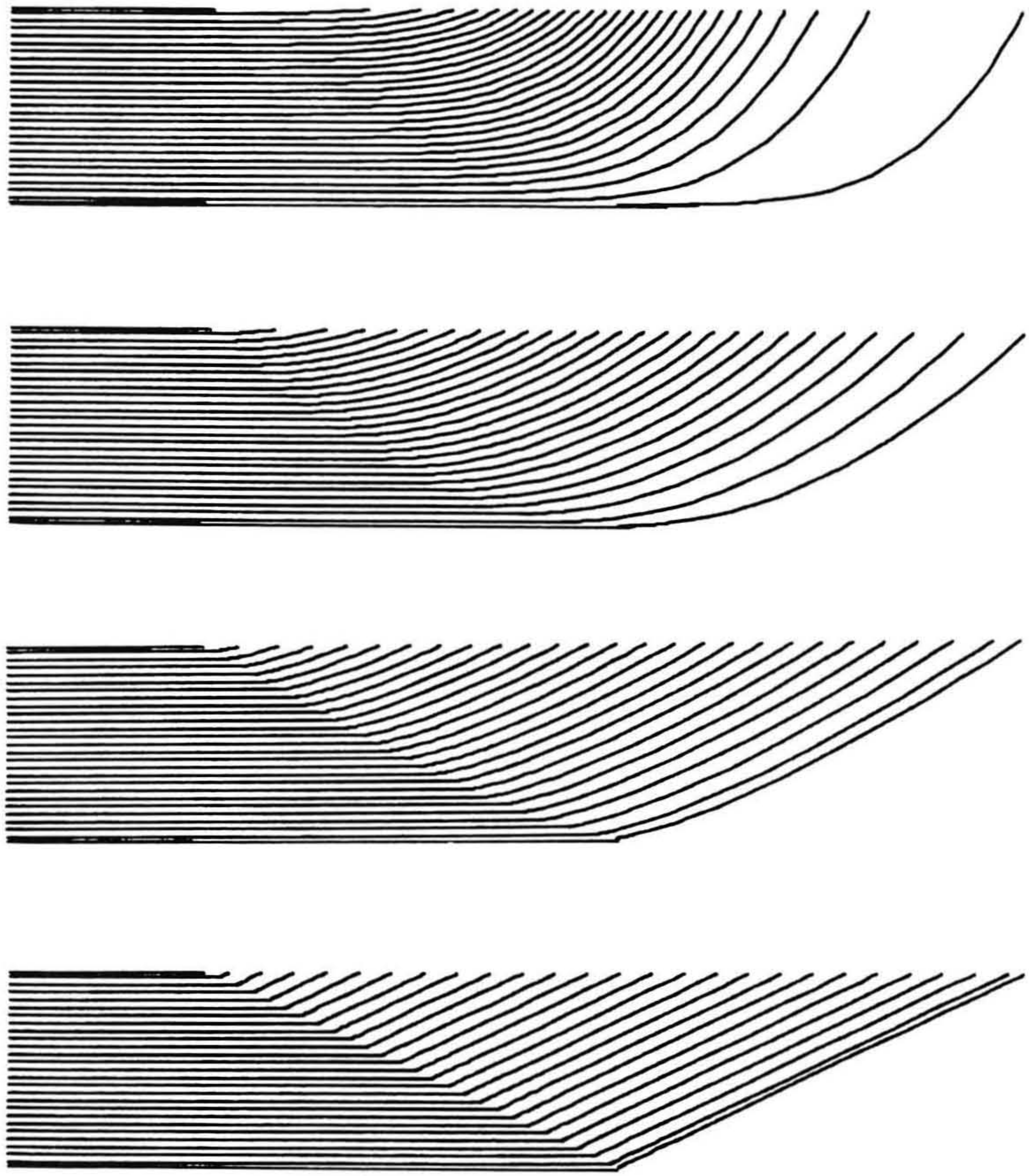

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Fig. 4. Side projections, y *vs*. z, of terminations for which $z_s(y) = (y/a)^p$, using p = 0.25, 0.50, 0.75, and 1.00, and with k = 2. With p = 1.0, the configuration assumes the form of the Lambertson-Coupland termination.

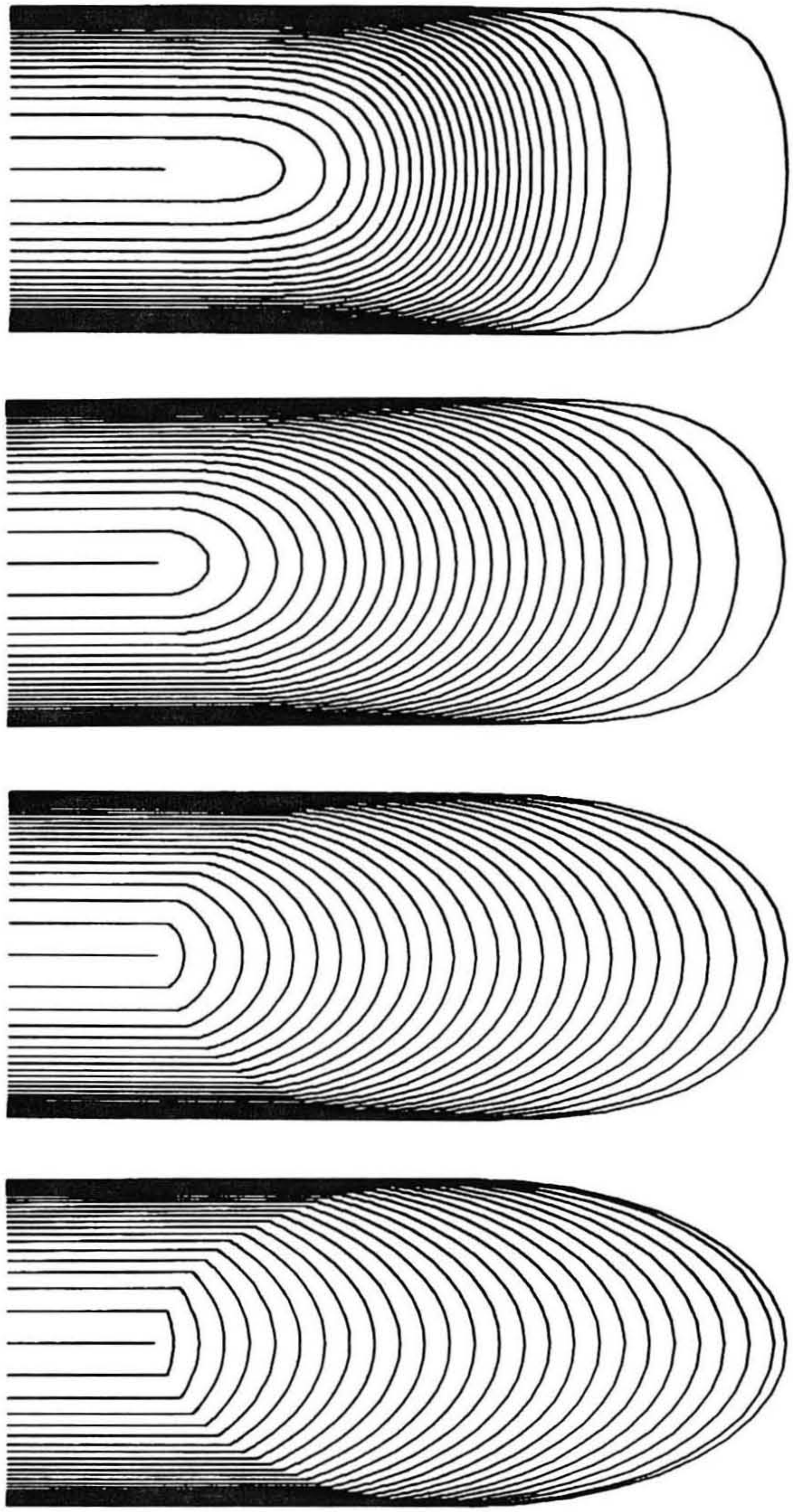

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Fig. 5. Top views, x vs. z, of the terminations for which side projections are shown in Fig. 4.

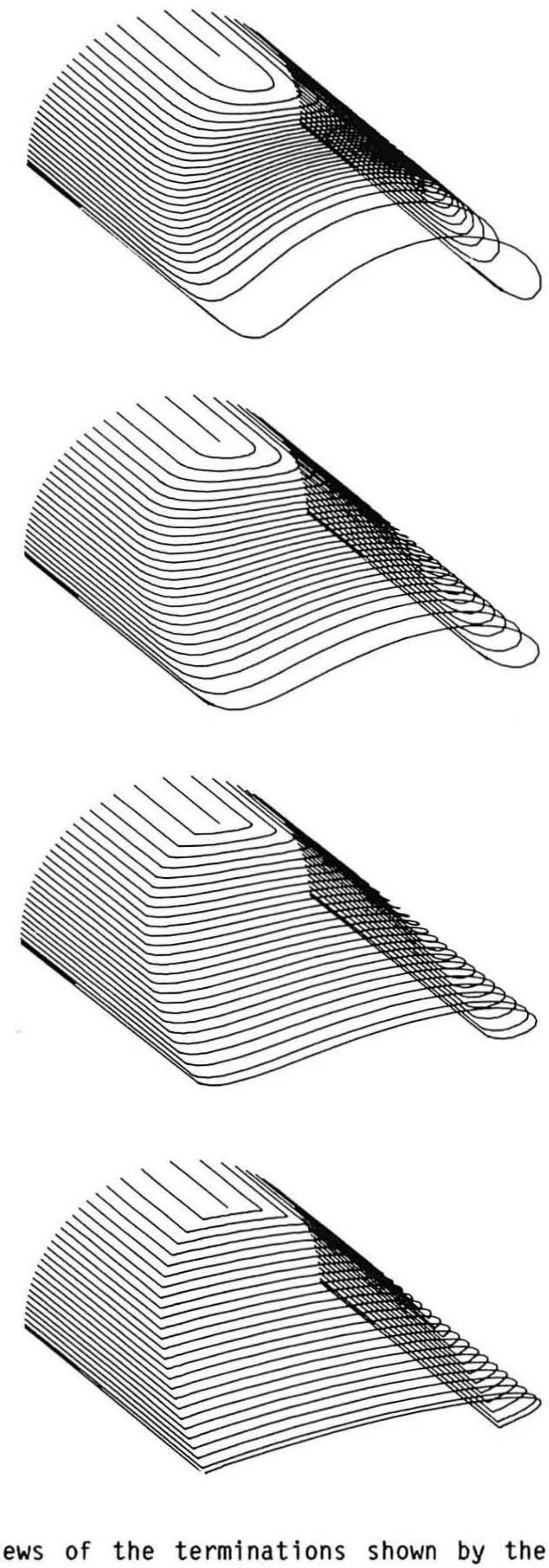

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Fig. 6. Isometric views of the terminations shown by the projected views of Figs. 4 and 5.

$x = \sqrt{a^2-y^2}$ vs. z, not a w-z developed view), and (c) an isometric view (Fig. 6). The appearance of a kink in the windings will be evident for $p > ½$ at the points of entry into the transition region. We are indebted to R. S. Digennaro of this Laboratory for the preparation of these diagrams.

3. Termination with a Circular (or Elliptical) Boundary in the Developed View

We noted previously that (as illustrated by Fig. 3) a termination of the type defined by

$$z_S(y) = - z_F(y) = f(y) = k(y/a)^{½}$$
$$= k(\cos \frac{w}{a})^{½}$$

results in a boundary for the termination that is approximately circular (or elliptical, depending upon the choice of the longitudinal scale factor k) in the developed view. Because of the freedom that may be exercised in the choice of the function f, however, one may, if it be desired, so choose this function that the transition will be bounded in this w-z development precisely by a circular or elliptical curve:

$$z_S = - z_F = b \left[(\frac{\pi}{2})^2 - (\frac{w}{a})^2 \right]^{½} = b \left[(\frac{\pi}{2})^2 - (\cos^{-1} \frac{y}{a})^2 \right]^{½}$$

for which the loci of the individual filaments in the y-z projection become given by

$$z(y;y_0) = b \left\{ \left[(\frac{\pi}{2})^2 - (\cos^{-1} \frac{y_0}{a})^2 \right]^{½} - \left[\cos^{-1} (\frac{y_0}{a})^2 - \cos^{-1} (\frac{y}{a})^2 \right]^{½} \right\} .$$

Such alternative forms for z_s are compared in Figs. 7 and 8. The form $z_s = - z_F = b \left[(\frac{\pi}{2})^2 - (\frac{w}{a})^2 \right]^{1/2}$ describes a circular boundary in the w-z developed plane if one makes the assignment b = a, as is illustrated by curve (b) in Fig. 7. A similar, but not strictly identical, curve (that also passes through the points z = 0, $w/a = \pm \frac{\pi}{2}$ and $z/a = \pm \frac{\pi}{2}$, w = 0) is obtained from the form $z_s = -z_F = k(\cos \frac{w}{a})^{1/2}$ with $k = \frac{\pi}{2}a$, as is illustrated by curve (a) of Fig. 7. Figure 8 depicts these same alternative forms for the boundary z_s in the y-z projection.

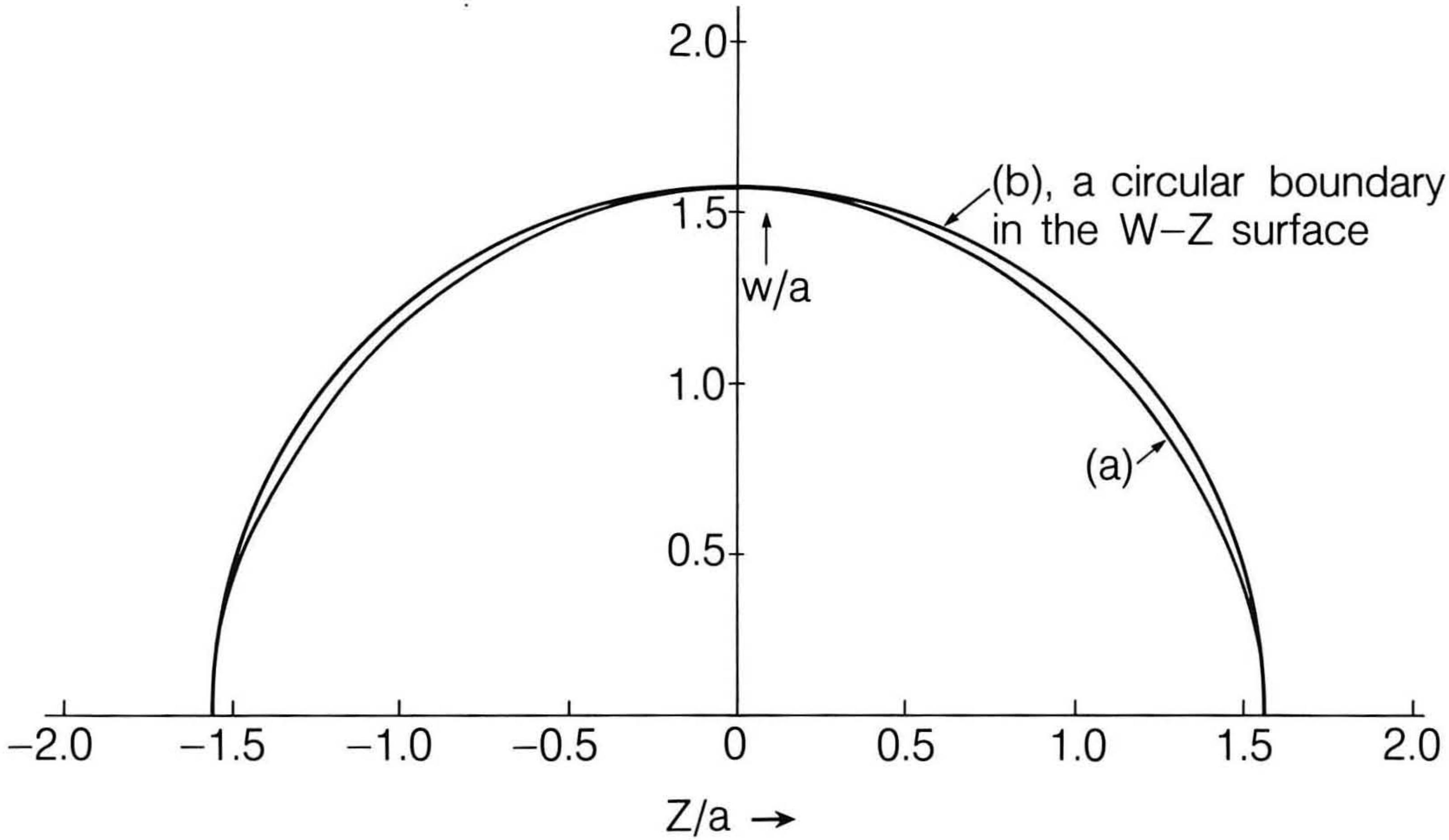

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Fig. 7. Plots in the w-z developed plane of the boundary to the transition region, for (a) $\frac{z_s}{a} = \frac{\pi}{2}\left(\cos\frac{w}{a}\right)^{1/2}$ and (b) for the circular form $\frac{z_s}{a} = \left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{w}{a}\right)^2\right]^{1/2}$, shown only for $w \geq 0$.

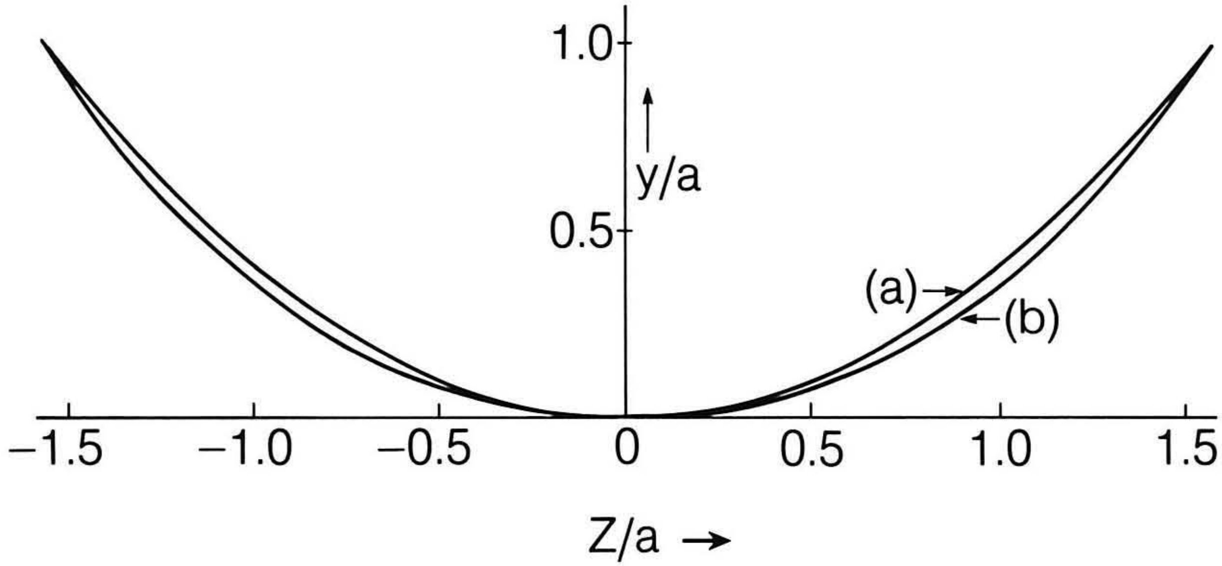

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Fig. 8. Plots, in the transverse y-z projection, of the boundaries z_s illustrated in Fig. 7. These boundaries are described in the present projection by (a) $\frac{z_s}{a} = \frac{\pi}{2} \left(\frac{y}{a}\right)^{1/2}$ and (b) $\frac{z_s}{a} = \left[\left(\frac{\pi}{2}\right)^2 - \left(\cos^{-1} \frac{y}{a}\right)^2 \right]^{1/2}$. The latter case is such as leads to a circular boundary in the w-z developed plane depicted in Fig. 7.

IV. Termination of Windings for Fields of Higher Multipolarity

A. Generalization of the Dipole Results:

Methods of terminating 2-D cos $m\phi$ windings so as to maintain the harmonic purity of the integrated internal field, are readily devised by extension of the procedure described earlier for termination of the cos ϕ windings of dipole magnets. As G. R. Lambertson has pointed out, this extension may be performed most directly by reference to the w-z development.

With a cos $m\phi$ winding present in the 2-D region, one chooses as the center for constructing the development of a w-z plane the point $\phi = \frac{\pi}{2m}$ (that constitutes one of the "poles" of the 2-D cos $m\phi$ winding) and writes the developed coördinate w as

$$w = a(\frac{\pi}{2m} - \phi) \quad \text{for} \; -\frac{\pi}{2m}a \leq w \leq \frac{\pi}{2m}a.*$$

An acceptable type of termination for a 2-D cos ϕ winding, expressible in the form

$$z = f(a \cos \frac{w_o}{a}) - f(a \cos \frac{w}{a} - a \cos \frac{w_o}{a}) \quad [\text{for } m = 1] \;,$$

becomes replaced by

$$z = f(a \cos \frac{mw_o}{a}) - f(a \cos \frac{mw}{a} - a \cos \frac{mw_o}{a}) \quad .$$

[See Fig. 10 as an illustration of such a termination of a cos 2ϕ (quadrupole) winding.] The boundary of the termination then becomes represented by

$$z_s = - z_F = f(a \cos \frac{mw}{a})$$
$$= f(a \sin m\phi) \; .$$

The termination obtained thus in the developed plane is simply a replica of the analogous termination for a cos ϕ winding, scaled down in the w direction

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*The value of y, of course, remains given by $y = a \sin \phi$.

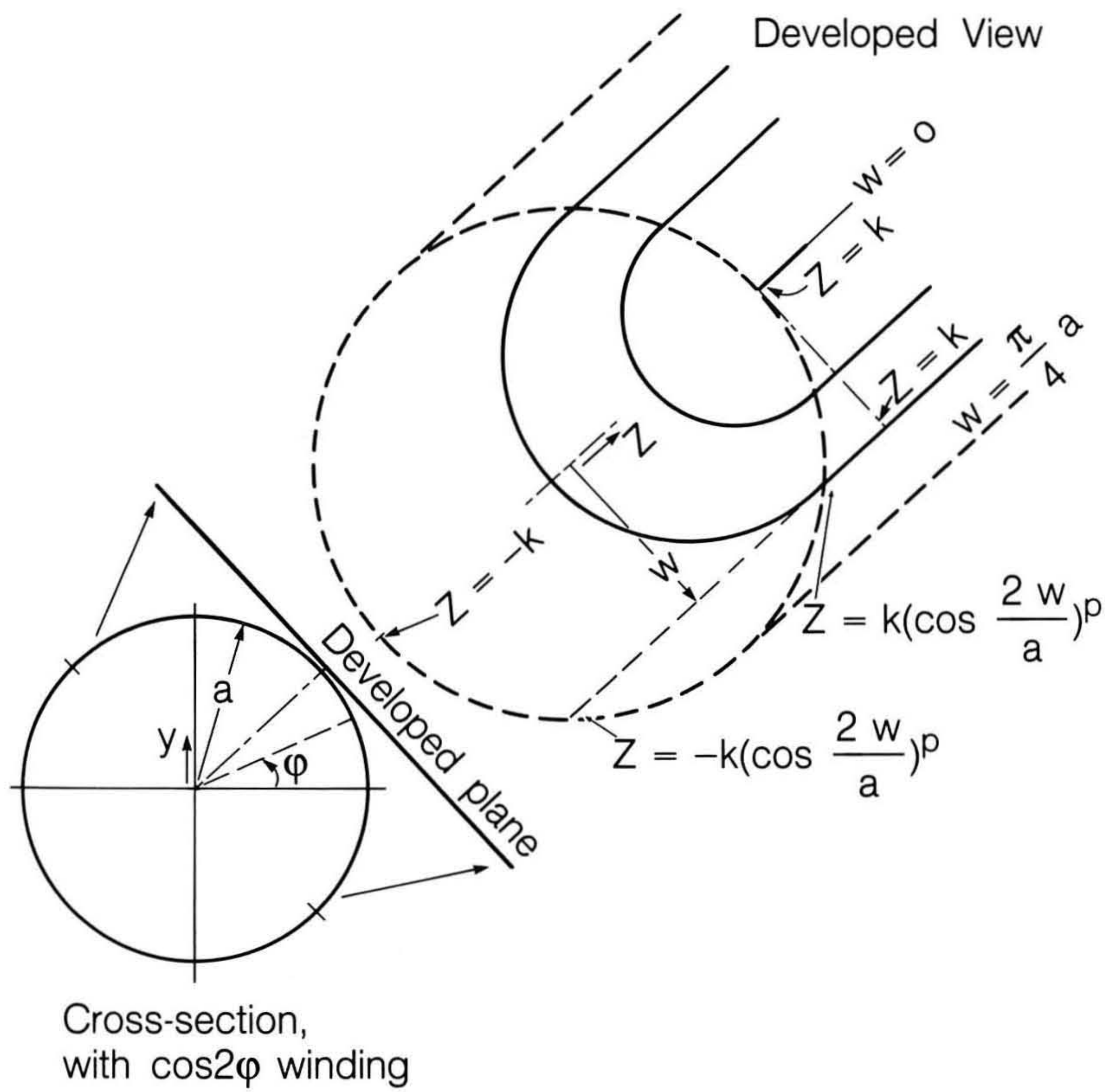

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Fig. 9. Illustration of the formation of a developed view for terminating cos 2ϕ windings so as to preserve the quadrupole character of the integrated internal field.

by the factor 1/m, and identical terminations also are to be constructed about the remaining poles at $\phi = 3\frac{\pi}{2m}, 5\frac{\pi}{2m}$, etc.

B. Confirmatory Calculation:

The results just cited for termination of cos $m\phi$ windings can be checked, if desired, by direct reference to the configuration as described in the w-z plane. To perform such a confirmatory calculation by reference to the w-z plane, we first note that the transformation between a $d\phi$ and w is distortion-free, so that in the w-z plane the reference J_z of the 2-D region is proportional to cos $m\phi$ and hence to $\sin \frac{mw}{a}$. We then accordingly need only to verify that in the w-z plane, the projected J_z component, upon integration at constant w (constant y) through the end region (and the approaches thereto), will lead to a result proportional to $\sin \frac{mw}{a}$.

Within the transition region of the w-z plane, the z-component of current becomes modified, from the value $J_0 \sin \frac{mw_0}{a}$ that applies at w_0 in the "straight" (2-D) region, through multiplication by $\left.\frac{\partial w_0}{\partial w}\right|_{z\ \text{const.}}$ -- to become $J_0 (\sin \frac{mw_0}{a}) \left[\frac{\partial w_0}{\partial w}\right]$. The integration at constant w through the transition portion of the end region then becomes (see Fig. 10)

$$\int_{z_F}^{z_S} \underbrace{J_0 (\sin \frac{mw_0}{a}) \left[\frac{\partial w_0}{\partial w}\right]}_{\text{at w const.}} dz = \int_{w_0=w}^{\frac{\pi}{2m}a} \underbrace{J_0 (\cos \frac{mw_0}{a}) \frac{\partial z}{\partial w}}_{\text{at w const.}} dw_0$$

and, when supplemented by the integral over the approach to the transition region provides the total

$$J_0 (\sin \frac{mw}{a}) \left[f(a) - f(a \cos \frac{mw}{a})\right]$$

$$+ \int_{w_0=w}^{\frac{\pi}{2m}a} m J_0 \sin \frac{mw}{a} \sin \frac{mw_0}{a} f'(a \cos \frac{mw}{a} - a \cos \frac{mw_0}{a})dw_0$$

$$= J_o \left(\sin \frac{m\pi w}{a}\right) \left[f(a) - f\left(a \cos \frac{m\pi w}{a}\right) \right]$$

$$+ J_o \sin \frac{m\pi w}{a} f\left(a \cos \frac{m\pi w}{a} - a \cos \frac{m\pi w_o}{a}\right) \Bigg|_{w_o=w}^{\frac{\pi}{2m}a}$$

$$= J_o \left(\sin \frac{m\pi w}{a}\right) \left[f(a) - f\left(a \cos \frac{m\pi w}{a}\right) \right]$$

$$+ J_o \sin \frac{m\pi w}{a} f\left(a \cos \frac{m\pi w}{a}\right) \quad [\text{since } f(0) = 0]$$

$$= J_o\, f(a) \left(\sin \frac{m\pi w}{a}\right)$$

and hence is found to be proportional to $\sin \frac{m\pi w}{a}$ (as required).

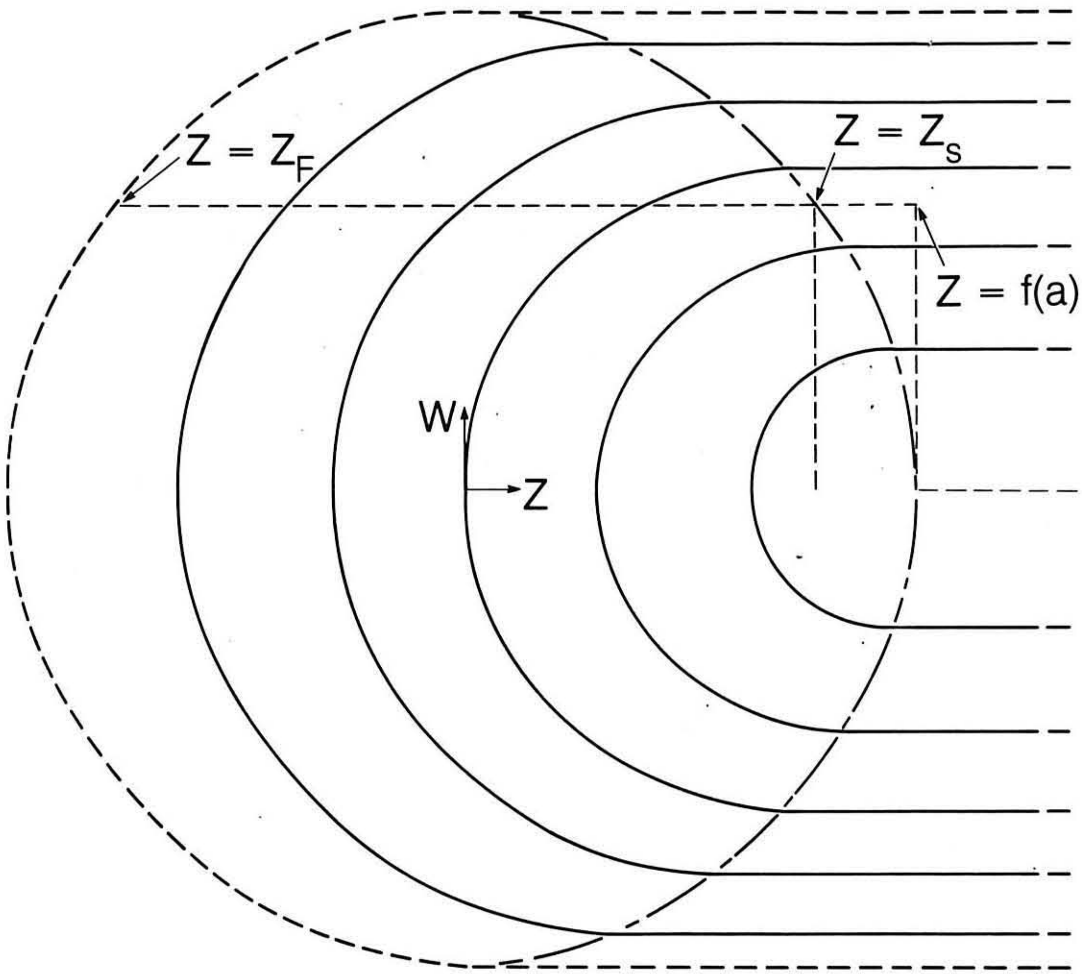

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Fig. 10. Sketch illustrating a path of integration, at constant w, in the w-z surface.

$$\text{At } z = z_S = f(a \cos \frac{mw}{a}),\ w_o = w;$$

$$\text{at } z = z_F = -f(a \cos \frac{mw}{a}),\ w_o = \frac{\pi}{2m}a.$$